4 Multiply and Divide Complex Numbers.

Multiplying complex numbers is similar to multiplying polynomials.

EXAMPLE 4 Find each product. Write each result in the form $a + bi$. **a.** $5i(4 - i)$ **b.** $(2 + 5i)(6 + 4i)$ **c.** $(3 + 5i)(3 - 5i)$

Strategy We will use the distributive property or the FOIL method to find the products.

Why We perform the indicated operations as if the complex numbers were polynomials with i as a variable.

Solution

a.
$$\begin{aligned} 5i(4 - i) &= 5i \cdot 4 - 5i \cdot i && \text{Distribute the multiplication by } 5i. \\ &= 20i - 5i^2 \\ &= 20i - 5(-1) && \text{Replace } i^2 \text{ with } -1. \\ &= 20i + 5 \\ &= 5 + 20i && \text{Write the result in the form } a + bi. \end{aligned}$$

b.
$$\begin{aligned} (2 + 5i)(6 + 4i) &= \overset{F}{12} + \overset{O}{8i} + \overset{I}{30i} + \overset{L}{20i^2} && \text{Use the FOIL method.} \\ &= 12 + 38i + 20(-1) && 8i + 30i = 38i. \text{ Replace } i^2 \text{ with } -1. \\ &= 12 + 38i - 20 \\ &= -8 + 38i && \text{Subtract: } 12 - 20 = -8. \end{aligned}$$

c.
$$\begin{aligned} (3 + 5i)(3 - 5i) &= \overset{F}{9} - \overset{O}{15i} + \overset{I}{15i} - \overset{L}{25i^2} && \text{Use the FOIL method.} \\ &= 9 - 25(-1) && \text{Add: } -15i + 15i = 0. \text{ Replace } i^2 \text{ with } -1. \\ &= 9 + 25 \\ &= 34 \end{aligned}$$

Written in the form $a + bi$, the product is $34 + 0i$.

Self Check 4 Find the product. Write each result in the form $a + bi$:
a. $4i(3 - 9i)$ **b.** $(3 - 2i)(5 - 4i)$
c. $(2 + 6i)(2 - 6i)$

Now Try Problems 43 and 45

In Example 4c, we saw that the product of the imaginary numbers $3 + 5i$ and $3 - 5i$ is the real number 34. We call $3 + 5i$ and $3 - 5i$ *complex conjugates* of each other.

Complex Conjugates The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates.**

Success Tip

To find the conjugate of a complex number, write it in standard form and change the sign between the real and imaginary parts from + to − or from − to +.

For example,

- $7 + 4i$ and $7 - 4i$ are complex conjugates.
- $5 - i$ and $5 + i$ are complex conjugates.
- $-6i$ and $6i$ are complex conjugates, because $-6i = 0 - 6i$ and $6i = 0 + 6i$.

In general, the product of the complex number $a + bi$ and its complex conjugate $a - bi$ is the real number $a^2 + b^2$, as the following work shows:

$$\begin{aligned} (a + bi)(a - bi) &= \overset{F}{a^2} - \overset{O}{abi} + \overset{I}{abi} - \overset{L}{b^2i^2} && \text{Use the FOIL method.} \\ &= a^2 - b^2(-1) && \text{Add: } -abi + abi = 0. \text{ Replace } i^2 \text{ with } -1. \\ &= a^2 + b^2 \end{aligned}$$

We can use this fact when dividing by a complex number. The process that we use is similar to rationalizing denominators.

EXAMPLE 5 Write each quotient in the form $a + bi$: **a.** $\frac{6}{7 - 4i}$ **b.** $\frac{3 - i}{2 + i}$

Strategy We will build each fraction by multiplying it by a form of 1 that uses the conjugate of the denominator.

Why This step produces a real number in the denominator so that the result can then be written in the form $a + bi$.

Solution

a. We want to build a fraction equivalent to $\frac{6}{7 - 4i}$ that does not have i in the denominator. To make the denominator, $7 - 4i$, a real number, we need to multiply it by its complex conjugate, $7 + 4i$. It follows that $\frac{7 + 4i}{7 + 4i}$ should be the form of 1 that is used to build $\frac{6}{7 - 4i}$.

$\frac{6}{7 - 4i} = \frac{6}{7 - 4i} \cdot \frac{7 + 4i}{7 + 4i}$ To build an equivalent fraction, multiply by $\frac{7 + 4i}{7 + 4i} = 1$.

$= \frac{42 + 24i}{49 - 16i^2}$ To multiply the numerators, distribute the multiplication by 6. To multiply the denominators, find $(7 - 4i)(7 + 4i)$.

$= \frac{42 + 24i}{49 - 16(-1)}$ Replace i^2 with -1. The denominator no longer contains i.

$= \frac{42 + 24i}{49 + 16}$ Simplify the denominator.

$= \frac{42 + 24i}{65}$ This notation represents the sum of two fractions that have the common denominator 65: $\frac{42}{65}$ and $\frac{24i}{65}$.

$= \frac{42}{65} + \frac{24}{65}i$ Write the result in the form $a + bi$.

b. We can make the denominator of $\frac{3 - i}{2 + i}$ a real number by multiplying it by the complex conjugate of $2 + i$, which is $2 - i$.

$\frac{3 - i}{2 + i} = \frac{3 - i}{2 + i} \cdot \frac{2 - i}{2 - i}$ To build an equivalent fraction, multiply by $\frac{2 - i}{2 - i} = 1$.

$= \frac{6 - 3i - 2i + i^2}{4 - i^2}$ To multiply the numerators, find $(3 - i)(2 - i)$. To multiply the denominators, find $(2 + i)(2 - i)$.

$= \frac{6 - 5i + (-1)}{4 - (-1)}$ Replace i^2 with -1. The denominator no longer contains i.

$= \frac{5 - 5i}{5}$ Simplify the numerator and the denominator.

$= \frac{5}{5} - \frac{5i}{5}$ Write each term of the numerator over the denominator, 5.

$= 1 - i$ Simplify each fraction.

Self Check 5 Write each quotient in standard form: **a.** $\frac{5}{4 - i}$ **b.** $\frac{5 - 3i}{4 + 2i}$

Now Try Problems 49 and 53

The Language of Algebra

Recall that the word *conjugate* was used in Chapter 8 when we rationalized the denominators of radical expressions such as $\frac{5}{\sqrt{6} - 1}$.

Caution

A common mistake is to replace i with -1. Remember, $i \neq -1$. By definition, $i = \sqrt{-1}$ and $i^2 = -1$.

Notation

It is acceptable to use $a - bi$ as a substitute for the form $a + (-b)i$. In Example 5b, we write $1 - i$ instead of $1 + (-1)i$.

Perform the operations. Write all answers in the form $a + bi$. See Example 4.

41. $3(2 - i)$

42. $9(-4 - 4i)$

43. $-5i(5 - 5i)$

44. $2i(7 + 2i)$

45. $(3 - 2i)(2 + 3i)$

46. $(3 - i)(2 + 3i)$

47. $(4 + i)(3 - i)$

48. $(1 - 5i)(1 - 4i)$

Write each quotient in the form $a + bi$. See Example 5.

49. $\dfrac{5}{2 - i}$

50. $\dfrac{26}{3 - 2i}$

51. $\dfrac{-4i}{7 - 2i}$

52. $\dfrac{5i}{6 + 2i}$

53. $\dfrac{2 + 3i}{2 - 3i}$

54. $\dfrac{2 - 5i}{2 + 5i}$

55. $\dfrac{4 - 3i}{7 - i}$

56. $\dfrac{4 + i}{4 - i}$